TEMPERATURE STRESSES IN A SLAB OF FINITE THICKNESS IN THE PRESENCE OF OSCILLATIONS OF THE AMBIENT TEMPERATURE ON ONE OR BOTH SIDES

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On the basis of a solution of the problem of thermoelasticity for a free slab of finite thickness the author presents formulas and graphs useful for practical calculations of the temperature stresses in the presence of ambient-temperature fluctuations on one or both sides and convective heat transfer at the surfaces.

The normal temperature stresses in a free slab of finite thickness in the presence of steady harmonic oscillations of the ambient temperature in the one-dimensional case, i.e., when the temperature of the slab varies only over the thickness z (Fig. 1), are calculated from the formula [1]

$$\sigma_{ii}(z,\ \tau) = \frac{\alpha E}{1-\nu}\left[-t(z,\ \tau)+t_1(\tau)+\frac{z}{H}\,t_2(t)\right], \tag{1}$$

where

$$t_1(\tau)=\frac{1}{H}\int\limits_{-H/2}^{H/2} t(z,\ \tau)\,dz,$$

$$t_2(\tau)=\frac{12}{H^2}\int\limits_{-H/2}^{H/2} t(z,\ \tau)\,z\,dz. \tag{2}$$

The subscript i takes the values x and y (coordinates in the plane of the slab).

Substituting in (1) the corresponding values of $t(z,\tau)$, $t_1(\tau)$, and $t_2(\tau)$ for unilateral and bilateral directions of heat flow, we determine the values of the nonstationary temperature stresses at any point of the slab in the presence of steady harmonic oscillations of the temperature of the medium, which are taken in the form of the real part of the complex variable

$$t = A_t \exp i\,\omega\tau = A_t(\cos\omega\tau + i\sin\omega\tau). \tag{3}$$

The temperature distribution function $t(z,\tau)$ is found [2, 3] from the solution of the differential equation of heat conduction

$$\frac{\partial t(z,\ \tau)}{\partial\tau} = a\,\frac{\partial^2 t(z,\ \tau)}{\partial z^2} \tag{4}$$

and the boundary conditions

$$\left.\begin{aligned} &H\,\frac{\partial t(z,\ \tau)}{\partial z}+\mathrm{Bi}_{\text{out}}\left[A_t^{\text{out}}\exp i\,\omega\tau - t\left(-\frac{H}{2},\ \tau\right)\right]=0 \\ &\qquad\text{for } z=-\frac{H}{2} \\ &H\,\frac{\partial t(z,\ \tau)}{\partial z}-\mathrm{Bi}_{\text{in}}\left[A_t^{\text{in}}\exp i\,\omega\tau - t\left(\frac{H}{2},\ \tau\right)\right]=0 \\ &\qquad\text{for } z=\frac{H}{2} \end{aligned}\right\}. \tag{5}$$

Substituting $t(z,\tau)$ into (2), we find the corresponding values of $t_1(\tau)$ and $t_2(\tau)$.

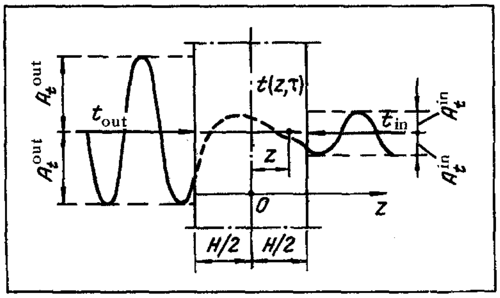

Fig. 1. Temperature distribution $t(z,\tau)$ over the thickness of an infinite slab in the presence of harmonic oscillations of the temperature on the left (t) and on the right (t_{in}).

After isolating the real part of the complex variable we write the expressions $t(z,\tau)$, $t_1(\tau)$, and $t_2(\tau)$ as follows: a) for one-sided heat flow from left to right (Fig. 1) with $t_{\text{in}} = 0$.

$$t^{\text{out}}(z,\ \tau) = A_t^{\text{out}}\frac{(ad+bf)\cos\omega\tau+(bd-af)\sin\omega\tau}{a^2+b^2}, \tag{6}$$

$$t_1^{\text{out}}(\tau) = A_t^{\text{out}}\,\frac{(am+bn)\cos\omega\tau+(bm-an)\sin\omega\tau}{\mu H(a^2+b^2)}, \tag{7}$$

$$t_2^{\text{out}}(\tau) = -A_t^{\text{out}}\frac{(ak+bl)\cos\omega\tau+(bk-al)\sin\omega\tau}{\mu H(a^2+b^2)}, \tag{8}$$

b) for one-sided heat flow from right to left with t = 0:

$$t^{\text{in}}(z,\ \tau) = A_t^{\text{in}}\frac{(ad'+bf')\cos\omega\tau+(bd'-af')\sin\omega\tau}{a^2+b^2}, \tag{9}$$

$$t_1^{\text{in}}(\tau) = A_t^{\text{in}}\frac{(am'+bn')\cos\omega\tau+(bm'-an')\sin\omega\tau}{\mu H(a^2+b^2)}, \tag{10}$$

$$t_2^{\text{in}}(\tau) = A_t^{\text{in}}\,\frac{(ak'+bl')\cos\omega\tau+(bk'-al')\sin\omega\tau}{\mu H(a^2+b^2)}, \tag{11}$$

where

$$\mu = \sqrt{\frac{\pi}{aT}}, \tag{12}$$

$$\begin{aligned} a = & -2\mu H(\mathrm{Bi}_{out} + \mathrm{Bi}_{in})\, \mathrm{sh}\,\mu H \sin\mu H + \\ & + (\mathrm{Bi}_{out}\mathrm{Bi}_{in} - 2\mu^2 H^2)\, \mathrm{sh}\,\mu H \cos\mu H - \\ & - (\mathrm{Bi}_{out}\mathrm{Bi}_{in} + 2\mu^2 H^2)\, \mathrm{ch}\,\mu H \sin\mu H, \end{aligned} \tag{13}$$

$$\begin{aligned} b = & 2\mu H(\mathrm{Bi}_{out} + \mathrm{Bi}_{in})\, \mathrm{ch}\,\mu H \cos\mu H + \\ & + (\mathrm{Bi}_{out}\mathrm{Bi}_{in} + 2\mu^2 H^2)\, \mathrm{sh}\,\mu H \cos\mu H + \\ & + (\mathrm{Bi}_{out}\mathrm{Bi}_{in} - 2\mu^2 H^2)\, \mathrm{ch}\,\mu H \sin\mu H, \end{aligned} \tag{14}$$

$$\begin{aligned} m = & \mathrm{Bi}_{out}\mathrm{Bi}_{in}(\mathrm{ch}\,\mu H \cos\mu H - 1) + \\ & + \mathrm{Bi}_{out}\,\mu H(\mathrm{sh}\,\mu H \cos\mu H - \mathrm{ch}\,\mu H \sin\mu H), \end{aligned} \tag{15}$$

$$\begin{aligned} n = & \mathrm{Bi}_{out}\mathrm{Bi}_{in}\, \mathrm{sh}\,\mu H \sin\mu H + \\ & + \mathrm{Bi}_{out}\,\mu H(\mathrm{ch}\,\mu H \sin\mu H + \mathrm{sh}\,\mu H \cos\mu H), \end{aligned} \tag{16}$$

$$\begin{aligned} m' = & \mathrm{Bi}_{out}\mathrm{Bi}_{in}(\mathrm{ch}\,\mu H \cos\mu H - 1) + \\ & + \mathrm{Bi}_{in}\,\mu H(\mathrm{sh}\,\mu H \cos\mu H - \mathrm{ch}\,\mu H \sin\mu H), \end{aligned} \tag{17}$$

$$\begin{aligned} n' = & \mathrm{Bi}_{out}\mathrm{Bi}_{in}\, \mathrm{sh}\,\mu H \sin\mu H + \\ & + \mathrm{Bi}_{in}\,\mu H(\mathrm{ch}\,\mu H \sin\mu H + \mathrm{sh}\,\mu H \cos\mu H), \end{aligned} \tag{18}$$

$$\begin{aligned} k = 6\mathrm{Bi}_{out} \Big[& \mu H(\mathrm{sh}\,\mu H \cos\mu H - \mathrm{ch}\,\mu H \sin\mu H) - \\ & - \frac{\mathrm{Bi}_{in}}{\mu H}(\mathrm{sh}\,\mu H \cos\mu H + \mathrm{ch}\,\mu H \sin\mu H) + \\ & + \mathrm{Bi}_{in}(\mathrm{ch}\,\mu H \cos\mu H + 1) + \\ & + 2(1 - \mathrm{ch}\,\mu H \cos\mu H)\Big], \end{aligned} \tag{19}$$

$$\begin{aligned} l = 6\mathrm{Bi}_{out} \Big[& \mu H(\mathrm{sh}\,\mu H \cos\mu H + \mathrm{ch}\,\mu H \sin\mu H) + \\ & + \frac{\mathrm{Bi}_{in}}{\mu H}(\mathrm{sh}\,\mu H \cos\mu H - \mathrm{ch}\,\mu H \sin\mu H) + \\ & + (\mathrm{Bi}_{in} - 2)\, \mathrm{sh}\,\mu H \sin\mu H\Big], \end{aligned} \tag{20}$$

$$\begin{aligned} k' = 6\mathrm{Bi}_{in} \Big[& \mu H(\mathrm{sh}\,\mu H \cos\mu H - \mathrm{ch}\,\mu H \sin\mu H) - \\ & - \frac{\mathrm{Bi}_{out}}{\mu H}(\mathrm{sh}\,\mu H \cos\mu H + \mathrm{ch}\,\mu H \sin\mu H) + \\ & + \mathrm{Bi}_{out}(\mathrm{ch}\,\mu H \cos\mu H + 1) + 2(1 - \mathrm{ch}\,\mu H \cos\mu H)\Big], \end{aligned} \tag{21}$$

$$\begin{aligned} l' = 6\mathrm{Bi}_{in} \Big[& \mu H(\mathrm{sh}\,\mu H \cos\mu H + \mathrm{ch}\,\mu H \sin\mu H) + \\ & + \frac{\mathrm{Bi}_{out}}{\mu H}(\mathrm{sh}\,\mu H \cos\mu H - \mathrm{ch}\,\mu H \sin\mu H) + \\ & + (\mathrm{Bi}_{out} - 2)\, \mathrm{sh}\,\mu H \sin\mu H\Big], \end{aligned} \tag{22}$$

$$\begin{aligned} d = \mathrm{Bi}_{out}\mathrm{Bi}_{in} \Big[& \mathrm{sh}\,\mu H\left(\frac{1}{2} - \zeta\right) \cos\mu H\left(\frac{1}{2} - \zeta\right) - \\ & - \mathrm{ch}\,\mu H\left(\frac{1}{2} - \zeta\right) \sin\mu H\left(\frac{1}{2} - \zeta\right)\Big] - \\ & - 2\mu H\,\mathrm{Bi}_{out}\, \mathrm{sh}\,\mu H\left(\frac{1}{2} - \zeta\right) \sin\mu H\left(\frac{1}{2} - \zeta\right), \end{aligned} \tag{23}$$

$$\begin{aligned} f = \mathrm{Bi}_{out}\mathrm{Bi}_{in} \Big[& \mathrm{sh}\,\mu H\left(\frac{1}{2} - \zeta\right) \cos\mu H\left(\frac{1}{2} - \zeta\right) + \\ & + \mathrm{ch}\,\mu H\left(\frac{1}{2} - \zeta\right) \sin\mu H\left(\frac{1}{2} - \zeta\right)\Big] + \\ & + 2\mu H\,\mathrm{Bi}_{out}\, \mathrm{ch}\,\mu H\left(\frac{1}{2} - \zeta\right) \cos\mu H\left(\frac{1}{2} - \zeta\right), \end{aligned} \tag{24}$$

$$\begin{aligned} d' = \mathrm{Bi}_{out}\mathrm{Bi}_{in} \Big[& \mathrm{sh}\,\mu H\left(\frac{1}{2} + \zeta\right) \cos\mu H\left(\frac{1}{2} + \zeta\right) - \\ & - \mathrm{ch}\,\mu H\left(\frac{1}{2} + \zeta\right) \sin\mu H\left(\frac{1}{2} + \zeta\right)\Big] - \\ & - 2\mu H\,\mathrm{Bi}_{out}\, \mathrm{sh}\,\mu H\left(\frac{1}{2} + \zeta\right) \sin\mu H\left(\frac{1}{2} + \zeta\right), \end{aligned} \tag{25}$$

$$\begin{aligned} f' = \mathrm{Bi}_{out}\mathrm{Bi}_{in} \Big[& \mathrm{sh}\,\mu H\left(\frac{1}{2} + \zeta\right) \cos\mu H\left(\frac{1}{2} + \zeta\right) + \\ & + \mathrm{ch}\,\mu H\left(\frac{1}{2} + \zeta\right) \sin\mu H\left(\frac{1}{2} + \zeta\right)\Big] + \\ & + 2\mu H\,\mathrm{Bi}_{in}\, \mathrm{ch}\,\mu H\left(\frac{1}{2} + \zeta\right) \cos\mu H\left(\frac{1}{2} + \zeta\right). \end{aligned} \tag{26}$$

Substituting (6)–(8) and (9)–(11) into (1), after transformations we obtain expressions for the nonstationary temperature stresses: for one-sided periodic heat flow from left to right:

$$\sigma_{ii}^{out}(z, \tau) = \frac{\alpha E}{1 - \nu} A_t^{out}(\Phi_1^{out}\cos\omega\tau + \Phi_2^{out}\sin\omega\tau), \tag{27}$$

for one-sided periodic heat flow from right to left:

$$\sigma_{ii}^{in}(z, \tau) = \frac{\alpha E}{1 - \nu} A_t^{in}\,[\Phi_1^{in}\cos(\omega\tau + \varphi_0) + \Phi_2^{in}\sin(\omega\tau + \varphi_0)], \tag{28}$$

where

$$\Phi_1^{out} = \frac{am + bn - \mu H(ad + bf) - \zeta(ak + bl)}{\mu H(a^2 + b^2)}, \tag{29}$$

$$\Phi_2^{out} = \frac{bm - an - \mu H(bd - af) - \zeta(bk - al)}{\mu H(a^2 + b^2)}, \tag{30}$$

$$\Phi_1^{in} = \frac{am' + bn' - \mu H(ad' + bf') + \zeta(ak' + bl')}{\mu H(a^2 + b^2)}, \tag{31}$$

$$\Phi_2^{in} = \frac{bm' - an' - \mu H(bd' - af') + \zeta(bk' - al')}{\mu H(a^2 + b^2)}. \tag{32}$$

The Φ are temperature stress functions φ_0 the phase shift of the heat flux oscillations on the left and right.

From (27) and (28), on the basis of the superposition principle, it is easy to obtain expressions for the temperature stresses when the effect of the medium is two-sided and symmetrical ($\varphi_0 = 0$):

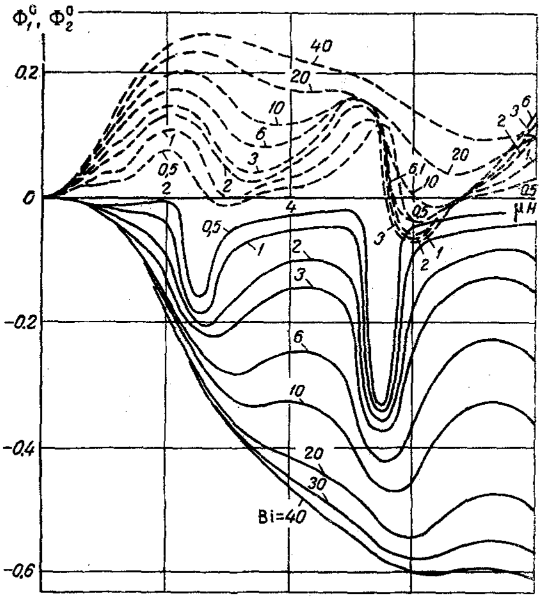

Fig. 2. Graphs of the temperature stress functions Φ_1^0, Φ_2^0 for $\zeta = 0.5$.

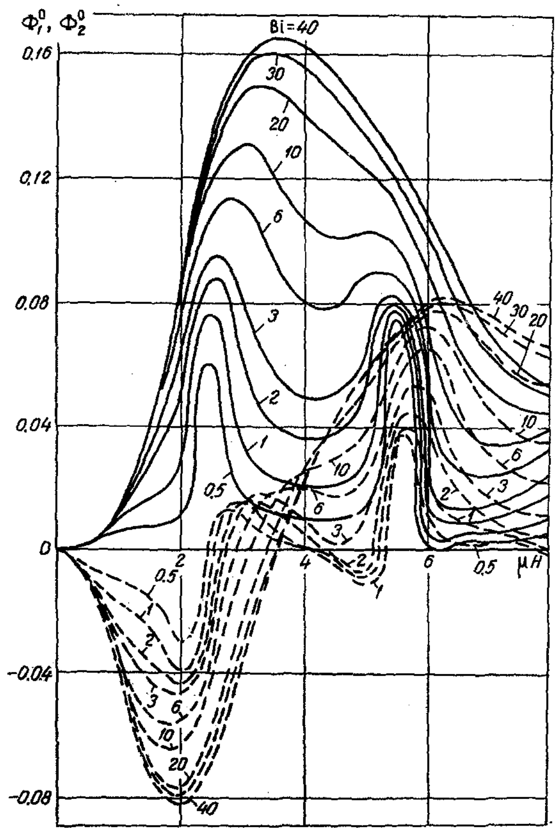

Fig. 3. Graphs of the temperature stress functions Φ_1^0, Φ_2^0 for $\zeta = 0$.

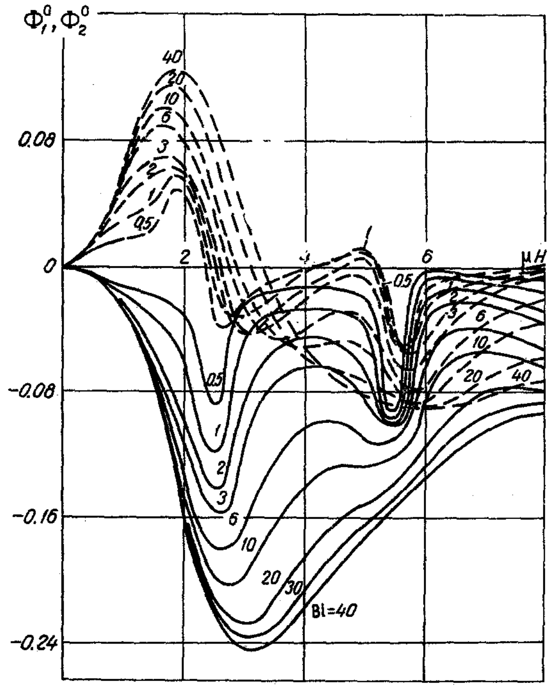

Fig. 4. Graphs of the temperature stress functions Φ_1^0, Φ_2^0 for $\zeta = 0.5$.

$$\sigma_{ii}^0 = \frac{\alpha E}{1-\nu} A_t \{[\Phi_1^0(\zeta) + \Phi_1^0(-\zeta)] \cos \omega\tau + $$
$$+ [\Phi_2^0(\zeta) + \Phi_2^0(-\zeta)] \sin \omega\tau\}, \quad (33)$$

where Φ_1^0 and Φ_2^0 are calculated from (29) and (30) with $Bi = Bi_{in}$.

Figures 2–4 show the graphs of the temperature stress functions Φ_1^0 (solid curves) and Φ_2^0 (dashed curves) in the presence of two-sided harmonic temperature oscillations for surface points with the coordinates $\zeta = \pm 0.5$ and in the center $\zeta = 0$ for $\alpha_{ex} = \alpha_{in} = 23.3$ W/m^2 · deg.

The maximum stresses at any point and the corresponding phases $\omega\tau_0$ are determined from the extremum condition

$$\frac{d\sigma_{ii}}{d\tau} = 0.$$

For one-sided action

$$\frac{d\sigma_{ii}}{d\tau} = \frac{\alpha E}{1-\nu} A_t \omega (-\Phi_1 \sin \omega\tau_0 + \Phi_2 \cos \omega\tau_0) = 0,$$

whence

$$\omega\tau_0 = \operatorname{arctg} \frac{\Phi_2}{\Phi_1}. \quad (34)$$

The maximum stresses are calculated by substituting values of $\omega\tau_0$ from (34) into (27) and (28). The procedure for two-sided action is analogous.

Example. To find the law of variation of the normal temperature stresses σ_{ii}^0 at the surfaces and in the center of an infinite slab of thickness H in the presence of two-sided harmonic oscillations of the daily air temperature of amplitude A_t for $\mu H = 3.1$ and $Bi = 21.8$.

From the graphs in Figs. 2–4 for the given values of μH and Bi we find Φ_1^0 and Φ_2^0 for the points

$$\zeta = -0.5, \ \Phi_1^0 = -0.36, \ \Phi_2^0 = 0.225;$$
$$\zeta = 0, \quad \Phi_1^0 = 0.151, \quad \Phi_2^0 = -0.02;$$
$$\zeta = 0.5, \quad \Phi_1^0 = -0.23, \ \Phi_2^0 = 0.$$

From (33) we find the expressions for the stresses at the surface of the slab,

$$\sigma_{ii}^0 = \frac{\alpha E}{1-\nu} \times$$
$$\times A_t [(-0.36-0.23) \cos \omega\tau + (0.225+0) \sin \omega\tau] =$$
$$= \frac{\alpha E}{1-\nu} A_t (-0.59 \cos \omega\tau + 0.225 \sin \omega\tau),$$

and at the center

$$\sigma_{ii}^0 = \frac{\alpha E}{1-\nu} \times$$
$$\times A_t [(0.151+0.151) \cos \omega\tau + (-0.02-0.02) \sin \omega\tau] =$$
$$= \frac{\alpha E}{1-\nu} A_t (0.302 \cos \omega\tau - 0.04 \sin \omega\tau).$$

NOTATION

t is the temperature; x, y, and z are coordinates; τ is time; H is the thickness of the plate; $\zeta = z/H$ is the relative coordinate; α is the coefficient of linear expansion; E is the modulus of elasticity of the material; ν is Poisson's ratio; a is the thermal diffusivity; λ is the thermal conductivity; α and α_{in} are the heat transfer coefficients at outside (left) and inside (right) surfaces; $Bi_{out} = \alpha_{out} H/\lambda$ and $Bi_{in} = \alpha_{in} H/\lambda$ denote the Biot number for outside (left) and inside (right) surfaces; A_t^{out} and A_t^{in} are the amplitudes of the temperature oscillations of the outside and inside air; $\omega = 1/T$ is the frequency of the harmonic oscillations; T is the period of the oscillations.

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